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A PERTURBATION ANALYSIS OF NON-LINEAR FREE FLEXURAL VIBRATIONS OF A CIRCULAR CYLINDRICAL SHELL

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Abstract—Non-linear free vibrations of a circular cylindrical shell are examined using Donnell's equations. A modal expansion is used for the normal displacement that satisfies the boundary conditions for the normal displacement exactly, but the boundary conditions for the in-plane displacements are satisfied approximately by an averaging technique. Galerkin technique is used to reduce the problem to a system of coupled non-linear ordinary differential equations for the modal amplitudes. These non-linear differential equations are solved for arbitrary initial conditions by using the multiple-time-scaling technique. Explicit values of the coefficients that appear in the forementioned Galerkin system of equations are given, in terms of non-dimensional parameters characterizing the shell geometry and material properties, for a three mode case, for which results for specific initial conditions are presented. A comparison of the results with those obtained in previous studies of the problem is presented and the discrepancies are discussed.

NOTATION

x	meridional length coordinate
β	circumferential angle
R	radius of the shell
α	x/R
h	thickness of shell
δ	h/R
L	length of cylinder
A	L/R, aspect ratio
u, v	inplane displacements
w	normal displacement
$\tilde{w}$	w/h
E, ν	Young's modulus, Poisson's ratio
D	$Eh^3/12(1-\nu)$
ρ_m	mass density
C	Eh
Φ	stress function
$\tilde{\Phi}$	Φ/R^2C
τ	$t\left[\dfrac{D}{\rho_m h R^4}\right]^{1/2}$ nondimensional time
K	R^2C/D
N_x, N_y	inplane stress resultants
A_{MN}, B_{MN}, A_{MO}	nondimensional modal amplitudes also redefined as q_1, q_2, q_3, etc.
M, N	modal numbers
$\dot{M}$	$M\pi/A$
C_1	$(\dot{M}^2+N^2)^{-2}$
C_2	$\dot{M}^{-4}$
C_3	N^{-4}
$\Omega_i^2, b_{ijk}, c_{ijkl}$	coefficients in Galerkin system of equations; listed in the Appendix for the three mode case
ε	a small parameter that characterizes the amplitudes of initial conditions on q_i
τ_m	multiple time scales

$\tilde{q}_{im}$ component of q_i multiplied by ε^m
ξ $M\pi/AN$ (ratio of circumferential wave length to axial wave length)
ψ $(N^2h/R)^2$

INTRODUCTION

NON-LINEAR effects have long been recognized to play an important role in determining the stability and response of thin shells and plates under dynamic and aeroelastic loads. Chu and Herrmann [1] first presented an analysis for flat plates, where they also demonstrated the consistency of neglecting inplane inertia terms in the study of non-linear flexural vibrations of a plate. Chu [2] later presented an analysis for circular cylindrical shells and also gave a solution using a single-mode approximation, which indicated that the non-linearity was always of the hardening type, i.e. the frequency increases with the amplitude of vibration. Nowinski [3] later confirmed the results of Chu [2]. On the other hand, some experiments by Evensen [4] and Olson [5] indicated that the non-linearity is of the softening type, i.e. the vibration frequency decreases with amplitude. Evensen [6] later pointed out that the mode shape assumed by Chu [2] does not satisfy the condition of continuity of the circumferential inplane displacement, v, (but satisfies the support conditions on the normal displacement, w) and conversely that the mode shape of Nowinski [3] satisfies the circumferential continuity condition on v, but violates the condition $w = 0$ at the support. Further, in the analyses of both Chu [2] and Nowinski [3], no restraints on the axial inplane-displacement, u, at the ends of the cylinder were imposed.

Evensen [6] gave an analysis for a simply-supported shell, with a two-mode approximation, which satisfies the circumferential continuity condition on v and the condition $w = 0$ at the supports. However, the moment-free condition at the simply supported end was violated to the order $0(A_{mn}^2)$ (where A_{mn} is the modal-amplitude) and further no constraint was imposed on the axial displacement u at the ends of the cylinder. Thus, as Evensen [6] himself points out, the assumed mode satisfies boundary conditions that actually lie somewhere between simply supported and clamped ends. It should also be pointed out that the two coupled modal equations derived by Evensen [6] contain both non-linear "inertia" terms as well as non-linear "elasticity" terms (the terminology is the same as in Bolotin [7]) in contrast to the equations of Chu and Nowinski [3] which contain only the nonlinear elasticity terms as in Duffing's equation. Using modes shapes similar to those of Evensen [6], Matsuzaki and Kobayashi [8] obtained similar modal equations for clamped and simply-supported shells. The results of Evensen [6] showed that, for the two-mode response case, the nature of non-linearity is dependent on the aspect ratio ξ (ratio of circumferential wavelength to axial wavelength); small values of ξ generally resulting in softening effects and large values of ξ resulting in hardening effects.

Dowell and Ventres [9] presented a set of modal equations, wherein the assumed modes satisfy the simple-support conditions for w exactly but both the circumferential continuity condition on v, as well as the restraint on the inplane-axial displacement were satisfied on the average. As is discussed in the following, this averaging method violates the continuity constraint on v to the order $0(A_{mn}^2)$. Dowell, however, did not present any solutions for his modal equations. However, such an averaging method has been shown to be a satisfactory approximation for a plate or ring by Bolotin [10], Fralich [11] and Dowell [12].

As for the solution techniques, the method of harmonic balance or so-called method of averaging (Ref. [13]) has been employed by Evensen [6] and Matsuzaki and Kobayashi [8] in studying the relation between the "natural" frequency and amplitude and the forced response near resonance, using only a two-mode approximation. Similar techniques were employed in the analysis of a circular ring by Evensen [14, 15], and Dowell [16] and by Olson and Fung [17] in the analysis of shell flutter. If one wishes to retain more axial or circumferential modes in the w expansion, the harmonic balance technique becomes very tedious. In this regard, numerical integration of equations of motion to obtain the time history of modal amplitudes have been suggested. However, due to wide variation of the linear natural frequencies of each mode, no reliable information is available concerning the numerical stability criteria on the time increments needed for the integration of non-linear modal equations.

The purpose of the present paper is to study the non-linear vibration behavior of the shell when the simple-support conditions on the normal displacement are satisfied exactly and the restraint on *both* the inplane displacements, u and v, is imposed, at least, in an integral average sense.

Thus, the starting point of the present paper is to derive the non-linear modal equations (using the averaging method similar to the approach of Dowell [9] to satisfy the boundary conditions), in a convenient non-dimensional form in terms of pertinent non-dimensional parameters characterizing the shell geometry and material properties. However, explicit expressions for the coefficients in the non-linear modal equations are given for only a three-mode case. A general solution procedure, for the free-vibration problem, when the shell is subjected to arbitrary initial conditions and responds in M axial modes and N circumferential modes, is given by assuming a uniformly valid asymptotic expansion using the method of multiple time-scales (Refs. [21, 22]). The relation between the natural frequency of each mode and its amplitude is derived theoretically for the general response problem. Finally, the solution procedure is illustrated by an example wherein the shell is initially disturbed in a specific mode and the response due to non-linear coupling is assumed to involve three modes. Comparisons of the present results are made with those obtained by Chu [2], Nowinski [3] and by Evensen [6]. A discussion of the noted discrepancies is presented.

BASIC EQUATIONS AND PROBLEM FORMULATION

The equations for non-linear free vibrations of a circular cylindrical shell, based on Donnell's shallow shell theory, have been shown by Chu [2], to be,

$$D\nabla^4 w+\rho_m h\frac{\partial^2 w}{\partial t^2}=\frac{1}{R}\frac{\partial^2\Phi}{\partial x^2}+\frac{1}{R^2}\left[\frac{\partial^2\Phi}{\partial\beta^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\Phi}{\partial x\partial\beta}\frac{\partial^2 w}{\partial x\partial\beta}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial\beta^2}\right] \tag{1}$$

and

$$\frac{\nabla^4\Phi}{C}=-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}+\frac{1}{R^2}\left[\left(\frac{\partial^2 w}{\partial x\partial\beta}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial\beta^2}\right] \tag{2}$$

where x is the meridional length coordinate, β the circumferential angle, R the radius of the shell, h is the thickness, w is the normal displacement (positive inwards), ∇^4 is the biharmonic operator and Φ is the inplane-stress function, such that

$$N_x=\frac{1}{R^2}\frac{\partial^2\Phi}{\partial\beta^2}\qquad N_\beta=\frac{\partial^2\Phi}{\partial x^2},\qquad N_{x\beta}=-\frac{1}{R}\frac{\partial^2\Phi}{\partial x\partial\beta}. \tag{3}$$

Further, the shell material is considered to be elastic, isotropic and the midsurface of the shell is used as a reference surface. Thus,

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad C = Eh. \tag{4}$$

Implicit in the derivation of equations (1) and (2) are: (a) the neglect of inplane inertia and consequently they are limited to purely bending motions, (b) only the predominant non-linear terms have been retained, and thus, for instance, the non-linearities in curvature strains have been ignored, (c) Donnell's approximation ($1/N^2 \ll 1$, where N is the circumferential wave number) has been used, which limits the analysis to $N \geq 5$, (d) the usual thin shell assumption $(h/R) \ll 1$ and the neglect of transverse-shear and rotary inertia efforts, (e) the non-linearities in the mid-plane stretching strain terms have been retained, using the stress–displacement relations,

$$\begin{aligned}
(1-\nu^2)\frac{N_x}{C} &= -\frac{\nu w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2R^2}\left(\frac{\partial w}{\partial \beta}\right)^2+\frac{\partial u}{\partial x}+\frac{\nu}{R}\frac{\partial v}{\partial \beta} = \frac{1-\nu^2}{R^2C}\frac{\partial^2 \Phi}{\partial \beta^2} \\
(1-\nu^2)\frac{N_\beta}{C} &= -\frac{w}{R}+\frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2R^2}\left(\frac{\partial w}{\partial \beta}\right)^2+\frac{1}{R}\frac{\partial v}{\partial \beta}+\nu\frac{\partial u}{\partial x} = \frac{1-\nu^2}{C}\frac{\partial^2 \Phi}{\partial x^2} \\
(1-\nu^2)\frac{N_{x\beta}}{C} &= 2(1-\nu)\left[\frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \beta}+\frac{1}{R}\frac{\partial u}{\partial \beta}+\frac{\partial v}{\partial x}\right] = -\frac{1-\nu^2}{RC}\frac{\partial^2 \Phi}{\partial x \partial \beta}.
\end{aligned} \tag{5}$$

It is assumed that the shell is simply supported, and thus, the boundary conditions on the normal displacement are,

$$w = 0 \quad \text{at } x = 0, L, \qquad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, L. \tag{6}$$

Following Bolotin [10], Fralich [11] and Dowell [9], the boundary conditions on the in-plane displacements are satisfied on an average, thus:

$$\int_0^{2\pi}\int_0^L \frac{\partial u}{\partial x}\partial x \partial \beta = \int_0^{2\pi} [u(L,\beta)-u(0,\beta)]\, d\beta = 0$$

$$\int_0^L\int_0^{2\pi} \frac{\partial v}{\partial \beta}\partial x \partial \beta = \int_0^L [v(x,2\pi)-v(x,0)]\, dx = 0$$

and

$$\int_0^{2\pi}\int_0^L N_{x\beta}\, \partial x \partial \beta = 0. \tag{7}$$

The first of these states that the axial displacements "on the average" are zero at $x = 0, L$; the second that the circumferential displacement v is continuous in the circumferential coordinate "on the average" and the last that the average shear is zero.

Using the non-dimensional variables,

$$\delta = \frac{h}{R}; \quad \alpha = \frac{x}{R}; \quad \tilde{w} = \frac{w}{h}; \quad \tilde{\Phi} = \frac{\Phi}{R^2C}; \quad \tau = t\left(\frac{D}{\rho_m h R^4}\right)^{1/2}; \quad K = \frac{R^2C}{D} \tag{8}$$

the equations (1) and (2) can be rewritten in non-dimensional form as,

$$\nabla^4\tilde{w}+\frac{\partial^2\tilde{w}}{\partial\tau^2} = \frac{K}{\delta}\frac{\partial^2\tilde{\Phi}}{\partial\alpha^2}+K\left[\frac{\partial^2\tilde{\Phi}}{\partial\beta^2}\frac{\partial^2\tilde{w}}{\partial\alpha^2}-2\frac{\partial^2\tilde{\Phi}}{\partial\alpha\partial\beta}\frac{\partial^2\tilde{w}}{\partial\alpha\partial\beta}+\frac{\partial^2\tilde{\Phi}}{\partial\alpha^2}\frac{\partial^2\tilde{w}}{\partial\beta^2}\right] \tag{9}$$

$$\nabla^4\tilde{\Phi} = -\delta\frac{\partial^2\tilde{w}}{\partial\alpha^2}+\delta^2\left[\left(\frac{\partial^2\tilde{w}}{\partial\alpha\partial\beta}\right)^2-\frac{\partial^2\tilde{w}}{\partial\alpha^2}\frac{\partial^2\tilde{w}}{\partial\beta^2}\right] \tag{10}$$

where, now,

$$\nabla^4 \equiv \frac{\partial\alpha^2}{\partial\alpha^4}+2\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+\frac{\partial^4}{\partial\beta^4}$$

and for convenience, one can also write equations (5)–(7) as

$$(1-\nu^2)\frac{\partial^2\tilde{\Phi}}{\partial\beta^2} = -\nu\delta\tilde{w}+\frac{\delta^2}{2}\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)^2+\frac{\nu\delta^2}{2}\left(\frac{\partial\tilde{w}}{\partial\beta}\right)^2+\delta\frac{\partial\tilde{u}}{\partial\alpha}+\nu\delta\frac{\partial\tilde{v}}{\partial\beta}$$

$$(1-\nu^2)\frac{\partial^2\tilde{\Phi}}{\partial\alpha^2} = -\delta\tilde{w}+\frac{\nu\delta^2}{2}\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)^2+\frac{\delta^2}{2}\left(\frac{\partial\tilde{w}}{\partial\beta}\right)^2+\delta\frac{\partial\tilde{v}}{\partial\beta}+\nu\delta\frac{\partial\tilde{u}}{\partial\alpha}$$

and, with the additional definitions,

$$N_x = C\frac{\partial^2\tilde{\Phi}}{\partial\beta^2}, \qquad N_y = C\frac{\partial^2\tilde{\Phi}}{\partial\alpha^2} \tag{11}$$

and

$$\tilde{w} = 0 \quad \text{at } \alpha = 0, A, \qquad \frac{\partial^2\tilde{w}}{\partial\alpha^2} = 0 \qquad \text{at } \alpha = 0, A \tag{12}$$

and

$$\int_0^{2\pi}\int_0^A \frac{\partial\tilde{u}}{\partial\alpha}\partial\alpha\partial\beta = \int_0^{2\pi}[u(A,\beta)-u(0,\beta)]\,d\beta = 0$$

$$\int_0^A\int_0^{2\pi}\frac{\partial\tilde{v}}{\partial\beta}\partial\alpha\partial\beta = \int_0^A[v(\alpha,2\pi)-v(0,2\pi)]\,dx = 0 \tag{13}$$

$$\int_0^A\int_0^{2\pi}\frac{\partial^2\tilde{\Phi}}{\partial\alpha\partial\beta}\partial\alpha\partial\beta = 0.$$

A modal expansion for w can be assumed as,

$$\tilde{w}(\alpha,\beta,\tau) = \sum_M\sum_N A_{MN}(\tau)\cos N\beta\sin\dot{M}\alpha+\sum_M\sum_N B_{MN}(\tau)\sin N\beta\sin\dot{M}\alpha \tag{14}$$

where $\dot{M} \equiv M\pi/A$. The above expansion satisfies the boundary conditions, equation (12) exactly, but the satisfaction of the in-plane displacement boundary conditions in an average sense as in equation (13), as will be shown in the following, involves errors of the order $0(A_{MN}^2)$. For purposes of illustration of the derivation of the Galerkin system of coupled ordinary differential equations for the time behavior of A_{MN} and B_{MN}, only the following terms are retained:

$$\tilde{w}(\alpha,\beta,\tau) = A_{MN}\cos N\beta\sin\dot{M}\alpha+B_{MN}\sin N\beta\sin\dot{M}\alpha+A_{MO}\sin\dot{M}\alpha. \tag{15}$$

By substituting equation (15) into the right hand side of equation (10), one can solve for $\tilde{\Phi}$ completely as,

$$\tilde{\Phi} = \tilde{\Phi}_h + \tilde{\Phi}_p \tag{15a}$$

where the subscripts h and p stand for "homogeneous" and "particular", respectively.

It can easily be shown that,

$$\begin{aligned}\tilde{\Phi}_p = {} & \delta C_2^{-1/2}[C_1 A_{MN} \sin \dot{M}\alpha \cos N\beta + C_1 B_{MN} \sin \dot{M}\alpha \sin N\beta + C_2 A_{MO} \sin \dot{M}\alpha] \\ & + \delta^2 C_2^{-1/2} C_3^{-1/2}\left[\frac{A_{MN}^2}{2^5}(C_2 \cos 2\dot{M}\alpha - C_3 \cos 2N\beta) + \frac{B_{MN}^2}{2^5}(C_2 \cos 2\dot{M}\alpha\right. \\ & + C_3 \cos 2N\beta) - \frac{1}{2^4} A_{MN} B_{MN} C_3 \sin 2N\beta - \frac{A_{MN}A_{MO}}{2}(C_3 \cos N\beta - C_1 \cos \dot{M}\alpha \cos N\beta) \\ & \left. - \frac{B_{MN}A_{MO}}{2}(C_3 \sin N\beta - C_1 \cos \dot{M}\alpha \sin N\beta)\right]\end{aligned} \tag{16}$$

where,

$$C_1 = (\dot{M}^2 + N^2)^{-2}, C_2 = \dot{M}^{-4} \quad \text{and } C_3 = N^{-4}. \tag{17}$$

A comparison of equation (16) with equation (3.5) of Ref. [9] by Dowell, who assumed the same modes as in equation (14) and satisfied similar boundary conditions as in equation (13) shows agreement in the solution for $\tilde{\Phi}_h$ except for the last two terms in equation (16), which are apparently missing in Ref. [9].

For the homogeneous part, keeping in mind the "average" inplane displacement boundary conditions (13), one can assume a simple function, as

$$\tilde{\Phi}_h = \frac{1}{C}(\tfrac{1}{2}\bar{N}_x\beta^2 + \tfrac{1}{2}\bar{N}_\beta\alpha^2 - \bar{N}_{x\beta}\alpha\beta) \tag{18}$$

where $\bar{N}_x$, $\bar{N}_\beta$ and $\bar{N}_{x\beta}$ are physically, the inplane restraint stresses generated at the ends of the shell, due to the prevention of the inplane displacements on the "average". Substituting for $\tilde{\Phi}$ from equation (15a) in (11) and using the boundary conditions (13), it can easily be shown that

$$\bar{N}_x\left(\frac{1-\nu^2}{C}\right) = \left(\frac{A_{MN}^2 + B_{MN}^2}{8}\right)\delta^2(C_2^{-1/2} + \nu C_3^{-1/2}) + \frac{\nu\delta}{M\pi}[(-1)^M - 1]A_{MO} + \frac{\delta^2 C_2^{-1/2}}{4}A_{MO}^2 \tag{19}$$

$$\bar{N}_\beta\left(\frac{1-\nu^2}{C}\right) = \left(\frac{A_{MN}^2 + B_{MN}^2}{8}\right)\delta^2(\nu C_2^{-1/2} + C_3^{-1/2}) + \frac{\nu\delta^2}{4}C_2^{-1/2}A_{MO}^2 + \frac{\nu^2\delta}{M\pi}[(-1)^M - 1]A_{MO} \tag{20}$$

$$\bar{N}_{x\beta} = 0. \tag{21}$$

Equations (19)–(21) agree with equations (3.10) of Ref. [9] by Dowell.

At this point it is worth mentioning that in substituting for $\partial\tilde{u}/\partial\alpha$ and $\partial\tilde{v}/\partial\beta$, in terms of $\tilde{w}$ and the derivatives of $\tilde{w}$ and $\tilde{\Phi}$ from equation (11), and using the "average" boundary conditions equation (13), some terms of the order $0(A_{MN}^2)$ and $0(B_{MN}^2)$ disappear in the process of integration and thus one can see that the averaging process incurs errors of $0(A_{MN}^2)$ in satisfying the inplane-displacement boundary conditions in an integrated "average" sense.

Once the functions $\tilde{\Phi}_p$ and $\tilde{\Phi}_h$ are solved for, as in equations (16) and (18)–(21), respectively, one can substitute for the total solution for $\tilde{\Phi}$ and for the assumed displacement function $\tilde{w}$ as given in equation (15), into equation (9), to obtain a single non-linear differential equation in time for the variables A_{MN}, B_{MN} and A_{MO}. From this equation, using the Galerkin technique and weighing in turn by each of the functions

$$\sin \dot{M}\alpha \cos N\beta \qquad \sin \dot{M}\alpha \sin N\beta \qquad \sin \dot{M}\alpha$$

and integrating over the shell midsurface, a system of three ordinary, coupled non-linear differential equations in time can be obtained for the unknowns A_{MN}, B_{MN} and A_{MO} thus:

$$\frac{\partial^2 q_i}{\partial \tau^2}+q_i\Omega_i^2+\sum_j\sum_k b_{ijk}q_jq_k+\sum_j\sum_k\sum_l C_{ijkl}q_jq_kq_l+0[q^5]=0 \qquad \text{for } i=1,2,3 \qquad (22)$$

where, for added convenience, the variables have been redefined as,

$$\begin{Bmatrix} A_{MN} \\ B_{MN} \\ A_{MO} \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}.$$

Explicit values of the coefficients Ω_i^2, b_{ijk} and C_{ijkl}, for the present three mode case under consideration, are given in the Appendix, in terms of the non-dimensional parameters of shell geometry and shell material properties. The coefficients b_{ijk} and C_{ijkl} given above, are found to agree with those in equations (3.11) and (3.12) of Ref. [9], where they are not listed explicitly, after appropriate reduction of Ref. [9] equations to the form of equation (22).

Even though explicit values are presented only for a three mode case, it can easily be seen that even for general (MXN) mode case as represented by equation (14), the final set of equations can still be represented by equation (22), where the dimension of the vector q increases accordingly. It can also be argued that since in the derivation of equation (1), *major* non-linear effects have been retained to $0(w^5)$, the corresponding equation (22) is correct to $0(q^5)$ in *major* non-linear effects in bending vibrations.

SOLUTION OF THE PROBLEM

The system of equation (22) will now be solved for any given initial conditions of the type,†

$$\begin{aligned} q_i &= \varepsilon\alpha_i \quad \text{at} \quad \tau = 0 \\ \frac{\partial q_i}{\partial \tau} &= \varepsilon\beta_i \quad \text{at } \tau = 0 \end{aligned} \qquad (23)$$

where ε is an arbitrarily small parameter, that defines the amplitudes of initial values on displacement and velocity. Correspondingly, one can define for the solution of equation (22), that

$$q_i(\tau) = \varepsilon\tilde{q}_i(\tau). \qquad (24)$$

† Note that the physical initial condition is $w_i = \varepsilon h\alpha_i$ and $\partial w_i/\partial\tau = \varepsilon h\beta_i$. The magnitudes of these are such that the shell undergoes large displacements, but small strains.

Upon substituting equation (24), equations (22) can be reduced to

$$\frac{\partial^2 \tilde{q}_i}{\partial \tau^2}+\Omega_i^2 \tilde{q}_i+\varepsilon \sum_j \sum_k b_{ijk}\tilde{q}_j\tilde{q}_k+\varepsilon^2 \sum_j \sum_k \sum_l C_{ijkl}\tilde{q}_j\tilde{q}_k\tilde{q}_l+0[\varepsilon^4] = 0. \tag{25}$$

Equations of the above type have been dealt with, by using the method of harmonic balance or of averaging by Krylov and Bogoliubov [18], Bogoliubov and Mitropolsky [13]. The method of two-variable expansion was also employed by Cole and Kevorkian [19] in studying single-degree of freedom systems. Morrison [20] discussed the close relationship of the above two approaches and pointed out the relative simplicity of the two variable expansion procedure. Kevorkian [21] later gave a detailed exposition of the two variable expansion procedure. The generalization of the procedure using multiple-time-scales and the validity of the asymptotic expansion procedure was also discussed by Nayfeh [22] for single degree-of-freedom systems.

When the number of degrees of freedom involved is large, as in equation (25), the method of averaging of Mitropolsky [13] becomes tedious. In what follows, a simpler asymptotic expansion procedure using the method of multiple-time-scales is given for the solution of equation (25). One can define multiple time-scales $\tau_0, \tau_1, \tau_2, \ldots$, etc., according as

$$\tau_m = (\varepsilon)^m t. \tag{26}$$

One can now assume that there exists a uniformly valid asymptotic solution for $\tilde{q}_i$ of the form,

$$\tilde{q}_i = \sum_{m=0}^{M} \varepsilon^m \tilde{q}_{im}(\tau_0, \tau_1, \ldots, \tau_m)+0[\varepsilon^{M+1}] \tag{27}$$

where it should be pointed out that $\tilde{q}_{im}$ are now functions of the independent time scales $\tau_0, \tau_1, \tau_2, \ldots$, etc. From equation (26), it can easily be seen that

$$\frac{\partial}{\partial \tau} = \sum_m \varepsilon^m \frac{\partial}{\partial \tau_m} \qquad \frac{\partial^2}{\partial \tau^2} = \sum_m \sum_n \varepsilon^m \varepsilon^n \frac{\partial^2}{\partial \tau_m \partial \tau_n} \tag{28}$$

substituting for $d^2/d\tau^2$ and $\tilde{q}_i$ from equations (28) and (27), respectively, in equation (24) and identifying terms multiplied by equal powers of ε, one can obtain the following system of equations:

terms with ε^0:

$$\frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_0^2}+\Omega_i^2 \tilde{q}_{i0} = 0 \tag{29}$$

terms with ε^1:

$$\frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_0^2}+2\frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_0 \partial \tau_1}+\Omega_i^2 \tilde{q}_{i1}+\sum_j \sum_k b_{ijk}\tilde{q}_{j0}\tilde{q}_{k0} = 0 \tag{30}$$

terms with ε^2:

$$\frac{\partial^2 \tilde{q}_{i2}}{\partial \tau_0^2}+\Omega_i^2 \tilde{q}_{i2}+2\frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_0 \partial \tau_1}+2\frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_0 \partial \tau_2}+\frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_1^2} + \sum_j \sum_k b_{ijk}(\tilde{q}_{j0}\tilde{q}_{k1}+\tilde{q}_{j1}\tilde{q}_{k0})+\sum_j \sum_k \sum_l C_{ijkl}\tilde{q}_{j0}\tilde{q}_{k0}\tilde{q}_{l0} = 0 \tag{31}$$

etc.

The boundary conditions, equation (23), can be likewise transformed as,

$$\tilde{q}_{i0} = \alpha_i, \tilde{q}_{i1} = \tilde{q}_{i2} = \cdots = 0 \quad \text{for } \tau_m = 0 \tag{32}$$

$$\frac{\partial \tilde{q}_{i0}}{\partial \tau_0} = \beta_i, \frac{\partial \tilde{q}_{i0}}{\partial \tau_1} + \frac{\partial \tilde{q}_{i1}}{\partial \tau_0} = 0, \qquad \frac{\partial \tilde{q}_{i0}}{\partial \tau_2} + \frac{\partial \tilde{q}_{i1}}{\partial \tau_1} + \frac{\partial \tilde{q}_{i2}}{\partial \tau_0} = 0 \tag{33}$$

etc., for $\tau_m = 0$.

SOLUTION FOR THE ZEROTH ORDER SYSTEM

The zeroth order system, i.e. equation (29), has the general solution

$$\tilde{q}_{i0} = A_i(\tau_1, \tau_2, \ldots)\,e^{\lambda\Omega_i\tau_0} + A_i^*(\tau_1, \tau_2, \ldots)\,e^{-\lambda\Omega_i\tau_0} \tag{34}$$

where $\lambda = \sqrt{-1}$, A_i is a complex quantity that is a function of the time scales $\tau_1, \tau_2, \ldots$, etc. and A_i^* is the complex conjugate of A_i. A_i and A_i^* can be determined from the initial conditions, (32) and (33).

SOLUTION FOR THE FIRST ORDER SYSTEM

After substituting for $\tilde{q}_{i0}$ from equation (32), the first order system, i.e. equation (30), can now be written as

$$\begin{aligned}\frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_0^2} + \Omega_i^2 \tilde{q}_{i1} = &-2\lambda\Omega_i \frac{\partial A_i}{\partial \tau_1} e^{\lambda\Omega_i\tau_0} + 2\lambda\Omega_i \frac{\partial A_i^*}{\partial \tau_1} e^{-\lambda\Omega_i\tau_0} \\ &-\sum_j \sum_k b_{ijk}[A_j A_k^*\, e^{\lambda\tau_0(\Omega_j-\Omega_k)} + A_k A_j^*\, e^{-\lambda\tau_0(\Omega_j-\Omega_k)} \\ &+ A_k A_j\, e^{\lambda\tau_0(\Omega_j+\Omega_k)} + A_k^* A_j^*\, e^{-\lambda\tau_0(\Omega_j+\Omega_k)}].\end{aligned} \tag{35}$$

In solving for equation (35), the terms on the right hand side which vary with frequency Ω_i must be suppressed, as otherwise these would lead to spurious resonance in the solution and hence would destroy its uniformity. This can be done, by setting,

$$\frac{\partial A}{\partial \tau_1} = \frac{\partial A^*}{\partial \tau_1} = 0. \tag{36}$$

Thus, the quantities A and A^* are not functions of τ_1 and hence it can be seen from equation (34), that the zeroth order solution $\tilde{q}_{i0}$ doesn't depend on the time-scale τ_1, a fact that can be traced to the presence of quadratic non-linear terms in the original set of equation (22). Now, equation (35) can be solved for $\tilde{q}_{i1}$ as,

$$\begin{aligned}\tilde{q}_{i1} = B_i\, e^{\lambda\Omega_i\tau_0} + B_i^*\, e^{-\lambda\Omega_i\tau_0} + \sum_j \sum_k [b'_{ijk}\, e^{\lambda\tau_0(\Omega_j-\Omega_k)} \\ + b''_{ijk}\, e^{\lambda\tau_0(\Omega_j+\Omega_k)} + b^{*\prime}_{ijk}\, e^{-\lambda\tau_0(\Omega_j-\Omega_k)} + b^{*\prime\prime}_{ijk}\, e^{-\lambda\tau_0(\Omega_j+\Omega_k)}]\end{aligned} \tag{37}$$

where $B_i = B_i(\tau_1, \tau_2 \ldots)$, depends on the initial values for $\tilde{q}_{i1}$, and the terms b'_{ijk} and b''_{ijk} can easily be seen to be,

$$b'_{ijk} = +\frac{b_{ijk} A_j A_k^*}{(\Omega_j - \Omega_k)^2 - \Omega_i^2}$$

no sum on j or k

$$b''_{ijk} = +\frac{b_{ijk}A_jA_k}{(\Omega_j+\Omega_k)^2-\Omega_i^2}$$

$$b^{*\prime}_{ijk}, b^{*\prime\prime}_{ijk} \equiv \text{conjugates of } b'_{ijk}, b''_{ijk}. \tag{38}$$

From the initial conditions, equations (32) and (33), it can easily be seen that

$$\tilde{q}_{i1} = 0 \quad \text{for } \tau_m = 0$$

and

$$\frac{\partial \tilde{q}_{i1}}{\partial \tau_0} = -\frac{\partial \tilde{q}_{i0}}{\partial \tau_1} = 0 \quad \text{for } \tau_m = 0 \tag{39}$$

since the zeroth order solution, $\tilde{q}_{i0}$, has been shown to be independent of τ_1. Using equation (39), the quantities B_i and B_i^* can be determined from equation (36), in terms of A_i and A_i^*.

SOLUTION FOR THE SECOND ORDER SYSTEM

Substituting for $\tilde{q}_{i0}$ and $\tilde{q}_{i1}$ from equations (34) and (37), respectively, the second order system, i.e. equation (31), can be reduced to,

$$\begin{aligned}
\frac{\partial^2 \tilde{q}_{i2}}{\partial \tau_0^2}+\Omega_i^2\tilde{q}_{i2} = {} & -2\left[\lambda\Omega_i \frac{\partial A_i}{\partial \tau_2} e^{\lambda\Omega_i\tau_0} - \lambda\Omega_i \frac{\partial A_i^*}{\partial \tau_2} e^{-\lambda\Omega_i\tau_0}\right] \\
& -2\left[\lambda\Omega_i \frac{\partial B_i}{\partial \tau_1} e^{\lambda\Omega_i\tau_0} - \lambda\Omega_i \frac{\partial B_i^*}{\partial \tau_1} e^{-\lambda\Omega_i\tau_0}\right] - \sum_j \sum_k b_{ijk}\Bigg[(A_j e^{\lambda\Omega_j\tau_0} \\
& + A_j^* e^{-\lambda\Omega_j\tau_0})\Bigg\{B_k e^{\lambda\Omega_k\tau_0} + B_k^* e^{-\lambda\Omega_k\tau_0} + \sum_l \sum_m (b'_{klm} e^{\lambda(\Omega_l-\Omega_m)\tau_0} \\
& + b''_{klm} e^{\lambda(\Omega_l+\Omega_m)\tau_0} + b^{*\prime} e^{-\lambda(\Omega_l-\Omega_m)\tau_0} + b^{*\prime\prime}_{klm} e^{-\lambda(\Omega_l+\Omega_m)\tau_0})\Bigg\} \\
& + (A_k e^{\lambda\Omega_k\tau_0} + A_k^* e^{-\lambda\Omega_k\tau_0})\Bigg\{B_j e^{\lambda\Omega_j\tau_0} + B_j^* e^{-\lambda\Omega_j\tau_0} \\
& + \sum_l \sum_m (b'_{jlm} e^{\lambda(\Omega_l-\Omega_m)\tau_0} + b''_{jlm} e^{\lambda(\Omega_l+\Omega_m)\tau_0} \\
& + b^{*\prime}_{jlm} e^{-\lambda(\Omega_l-\Omega_m)\tau_0} + b^{*\prime\prime}_{jlm} e^{-\lambda(\Omega_l+\Omega_m)\tau_0})\Bigg\}\Bigg] \\
& - \sum_j \sum_k \sum_l C_{ijkl}(A_j e^{\lambda\Omega_j\tau_0} + A_j^* e^{-\lambda\Omega_j\tau_0})(A_k e^{\lambda\Omega_k\tau_0} \\
& + A_k^* e^{-\lambda\Omega_k\tau_0})(A_l e^{\lambda\Omega_l\tau_0} + A_l^* e^{-\lambda\Omega_l\tau_0}).
\end{aligned} \tag{40}$$

Grouping terms on the right hand side that vary with frequency Ω_i, equation (40) can be rewritten as

$$\begin{aligned}
\frac{\partial^2 \tilde{q}_{i2}}{\partial \tau_0^2}+\Omega_i^2\tilde{q}_{i2} = {} & -2\Bigg(\lambda\Omega_i \frac{\partial A_i}{\partial \tau_2} + \lambda\Omega_i \frac{\partial B_i}{\partial \tau_1} \\
& + A_i \sum_l \gamma_l A_l A_l^*\Bigg) e^{\lambda\Omega_i\tau_0} + 2\Bigg(\lambda\Omega_i \frac{\partial A_i^*}{\partial \tau_2} \\
& + \lambda\Omega_i \frac{\partial B_i^*}{\partial \tau_1} + A_i^* \sum_l \gamma_l A_l A_l^*\Bigg) e^{-\lambda\Omega_i\tau_0} \\
& + \sum_r p_r e^{\lambda\Omega_r\tau_0} + \sum_r p_r^* e^{-\lambda\Omega_r\tau_0}.
\end{aligned} \tag{41}$$

In the above, Ω_r stands for the combinations,

$$\Omega_r = \pm\Omega_j \pm \Omega_k \pm \Omega_l \tag{42}$$

such that $\Omega_r \neq \Omega_i$ and thus the terms on the right hand side of equation (41) which vary with frequency Ω_r, would not produce any spurious resonance and hence would not alter the uniformity of the solution. The explicit expressions for p_r are very lengthy and hence are not recorded here. After some manipulation, it can be shown that the coefficients γ_l associated with the terms varying with frequency Ω_i on the right hand side of equation (41) can be shown to be,

$$\gamma_l = \tfrac{3}{2}C_{iiii} + \tfrac{1}{2}\sum_k (2b_{iik}b'_{kii} + 2b_{iki}b'_{kii} + b_{iik}b''_{kii} + b_{iki}b''_{kii}) \quad \text{for } l = i \tag{43a}$$

$$\begin{aligned}\gamma_l = \tfrac{1}{2}(C_{iill} + C_{ilil} + C_{illi}) + \tfrac{1}{2}\sum_k (&2b_{iik}b'_{kll} + b_{ilk}b'_{kil} \\ &+ b_{ilk}b'_{kli} + b_{ilk}b''_{kil} + b_{ilk}b''_{kli} + 2b_{iki}b'_{kil} + b_{ikl}b'_{kli} \\ &+ b_{ikl}b'_{kil} + b_{ikl}b''_{kli} + b_{ikl}b''_{kil}).\end{aligned} \tag{43b}$$

Spurious resonance in the solution of equation (41) can be eliminated by setting

$$\lambda\Omega_i \frac{\partial A_i}{\partial \tau_2} + A_i \sum_l \gamma_l A_l A_l^* = 0 \tag{44a}$$

and

$$\frac{\partial B_i}{\partial \tau_1} = 0. \tag{44b}$$

The system of equation (44a) can easily be solved by letting

$$A_n = \xi_n \, e^{\lambda\theta_n} \tag{45}$$

where ξ_n is a real quantity. Substituting equation (45) into equation (44) yields,

$$\lambda\Omega_i \left(\frac{\partial \xi_i}{\partial \tau_2} + \lambda \xi_i \frac{\partial \theta_i}{\partial \tau_2} \right) + \xi_i \sum_l \gamma_l \xi_l^2 = 0 \tag{46}$$

Separating real and imaginary parts in equation (46), one can obtain

$$\frac{\partial \xi_i}{\partial \tau_2} = 0 \tag{47}$$

and

$$\frac{\partial \theta_i}{\partial \tau_2} = \frac{1}{\Omega_i} \sum_l \gamma_l \xi_l^2. \tag{48}$$

Equation (47) can be solved as,

$$\xi_i = \tilde{\xi}_i(\tau_3, \tau_4, \ldots). \tag{49}$$

Thus, the right hand side of equation (48) is independent of τ_2 and hence, one can write

$$\theta_i = \frac{1}{\Omega_i}\left[\sum_{k=1}^{N} \gamma_k \xi_k^2\right]\tau_2 + \theta_{i0}(\tau_3, \ldots, \tau_m). \tag{50}$$

Using equations (49) and (50), equation (45) can be written as,

$$A_n = \tilde{A}_n \exp\left[\left(\frac{1}{\Omega_n}\sum_{k=1}^{N} \gamma_k \xi_k^2\right)\tau_2\right] \tag{51}$$

where

$$\tilde{A}_n(\tau_3, \tau_4, \ldots) = \xi_n \, e^{\lambda\theta_{n0}}.$$

Finally, one can now solve equation (41) for $\tilde{q}_{i2}$ as

$$\tilde{q}_{i2} = C_i \, e^{\lambda\Omega_i\tau_0} + C_i^* \, e^{-\lambda\Omega_i\tau_0} + \sum_{r=1}^{N}\left(\frac{p_r \, e^{\lambda\Omega_r\tau_0} + p_r^* \, e^{-\lambda\Omega_r\tau_0}}{\Omega_r^2 - \Omega_i^2}\right) \tag{52}$$

where c_i are to be determined from the initial conditions, equation (32), i.e.

$$\left.\begin{aligned} \tilde{q}_{i2} &= 0 \quad \text{for } \tau_m = 0 \\ \frac{\partial\tilde{q}_{i2}}{\partial\tau_0} &= -\left(\frac{\partial\tilde{q}_{i0}}{\partial\tau_2} + \frac{\partial\tilde{q}_{i1}}{\partial\tau_1}\right) \quad \text{for } \tau_m = 0. \end{aligned}\right. \tag{53}$$

Using equation (51), the zeroth order solution, equation (34), can be rewritten as,

$$\begin{aligned} \tilde{q}_{i0} = {} & \tilde{A}_i \, e^{\lambda\Omega_i\tau_0} \exp\left(\frac{\tau_2}{\Omega_i}\sum_{l=1}^{N} \gamma_l \tilde{A}_l^* \tilde{A}_l\right) \\ & + \tilde{A}_i^* \, e^{-\lambda\Omega_i\tau_0} \exp\left(\frac{\tau_2}{\Omega_i}\sum_{l=1}^{N} \gamma_l \tilde{A}_l \tilde{A}_l^*\right). \end{aligned} \tag{54}$$

Using equations (54), (37) and (52) in equations (24) and (27), the total solution can now be written as

$$\begin{aligned} q_i(\tau) = {} & \varepsilon\tilde{q}_{i0} + \varepsilon^2\tilde{q}_{i1} + \varepsilon^3\tilde{q}_{i2} + 0[\varepsilon^4] \\ = {} & \varepsilon[\tilde{A}_i \, e^{\lambda\tilde{\Omega}_i\tau} + \tilde{A}_i^* \, e^{-\lambda\tilde{\Omega}_i\tau}] + \varepsilon^2[B_i \, e^{\lambda\Omega_i\tau} + B_i^* \, e^{-\lambda\Omega_i\tau} \\ & + \sum_j \sum_k (b'_{ijk} \, e^{\lambda\tau(\Omega_j - \Omega_k)} + b^{*\prime}_{ijk} \, e^{-\lambda\tau(\Omega_j - \Omega_k)} + b''_{ijk} \, e^{\lambda\tau(\Omega_j + \Omega_k)} + b^{*\prime\prime}_{ijk} \, e^{-\lambda\tau(\Omega_j + \Omega_k)})] \\ & + \varepsilon^3\left[C_i \, e^{\lambda\Omega_i\tau} + C_i^* \, e^{-\lambda\Omega_i\tau} + \sum_{r=1}^{N}\left(\frac{p_r \, e^{\lambda\Omega_r\tau_0} + p_r^* \, e^{-\lambda\Omega_r\tau_0}}{\Omega_r^2 - \Omega_i^2}\right)\right] + 0[\varepsilon^4] \end{aligned} \tag{55}$$

where

$$\tilde{\Omega}_i = \Omega_i\left[1 + \varepsilon^2 \frac{1}{\Omega_i^2}\sum_{k=1}^{N} \gamma_k \tilde{A}_k \tilde{A}_k^*\right]. \tag{56}$$

Equation (55) indicates that, because of the non-linear coupling between the modes, even though the initial conditions on the shell correspond only to a single mode, other modes would also be excited. Further it can be seen that if the principal response of the excited mode is of order $0(\varepsilon)$, the other modes are excited to order $0(\varepsilon^2)$. Equation (56), on the other hand, indicates the effect of non-linearity on the frequencies of natural oscillations of each mode. The nature of the non-linearity (whether hand-spring type or soft-spring type) depends on whether the quantity

$$\sum_{k=1}^{N} \gamma_k \tilde{A}_k \tilde{A}_k^*$$

is positive or negative. From equation (43), it can be seen that γ_k depends on the non-linearity coefficients b_{ijk} and C_{ijkl}. For the present 3 mode response case, if M is even, as can be seen from the Appendix, all b_{ijk} vanish; and further the coefficients C_{1111}, C_{2222}, C_{2211} and C_{1122}, C_{1133} and C_{2233} are always positive, while the sign of the co-efficients C_{3311}, C_{3322}, C_{3333}, would depend upon the parameters $(M\pi/A)$, N and h/R. Thus, if the initial conditions on the shell are such that only either one of the first two modes is initially excited, with M even, it can be seen from equation (43) that γ_l is positive and a hard spring type non-linearity would result. If the shell is excited initially in the third mode, the nature of non-linear behavior can only be decided based on the parameters $(M\pi/A)$ and δ.

If on the other hand, M is odd, the coupling between the first two modes and the third mode exists in the form of non-vanishing b_{ijk} and consequently, the sign of γ_k in equation (43) can only be divided for particular values of $(M\pi/A)$, N and δ. To put the discussion that follows in the proper perspective, a specific example is given.

RESULTS FOR PARTICULAR INITIAL CONDITIONS

Let the shell be subjected to the initial velocity conditions, as

$$\tilde{W}(\alpha,\beta,0) = 0\frac{\partial \tilde{w}}{\partial \tau}(\alpha,\beta,0) = \varepsilon\beta_1\Omega_1 \sin \dot{M}\alpha \cos N\beta = \varepsilon\beta_1\Omega_1 \sin \dot{M}\alpha \cos\left(\frac{\dot{M}\beta}{\xi}\right) \tag{57}$$

where

$$\xi = \frac{M\pi}{AN} = \left(\frac{M}{L}\right)\left(\frac{\pi R}{N}\right) = \frac{\text{Circumferential wave length}}{\text{Axial wave length}}. \tag{58}$$

For purposes of illustration, consider the response of the shell to be represented by,

$$\tilde{W}(\alpha,\beta,\tau) = \tilde{q}_1(\tau)\sin \dot{M}\alpha \cos\left(\frac{\dot{M}\beta}{\xi}\right) + \tilde{q}_2(\tau)\sin \dot{M}\alpha \sin\left(\frac{\dot{M}\beta}{\xi}\right) + \tilde{q}_3(\tau)\sin \dot{M}\alpha. \tag{59}$$

It can be seen from equation (55) and using initial conditions, equation (57), the solution of the zeroth order system is given by,

$$\tilde{q}_{10} = \beta_1 \sin\tilde{\Omega}_1\tau, \qquad \tilde{q}_{20} = \tilde{q}_{30} = 0 \tag{60}$$

where

$$\tilde{\Omega}_1 = \Omega_1\left[1+\varepsilon^2\frac{\beta_1^2\gamma_1}{4\Omega_1^2}\right] \tag{61}$$

with

$$\Omega_1^2 = \left[\dot{M}^2 + \left(\frac{\dot{M}}{\xi}\right)^2\right]^2 + \frac{12(1-\nu^2)R^2(\dot{M})^4}{h^2}\left[\dot{M}^2 + \left(\frac{\dot{M}}{\xi}\right)^2\right]^{-2}. \tag{62}$$

Further, as can easily be seen from equation (43a) and the Appendix,

$$\gamma_1 = \frac{3}{2}C_{1111} + \frac{1}{2}\left(-\frac{2b_{113}b_{311}}{\Omega_3^2} + \frac{b_{113}b_{311}}{4\Omega_1^2 - \Omega_3^2}\right). \tag{63}$$

From equation (63), it can be seen that when M is even,

$$\gamma_1 = \tfrac{3}{2}C_{1111} \tag{64}$$

which is always positive (since C_{1111} is) and hence the non-linear natural frequency always increases with amplitude, as indicated by equation (61). However, the sign of the second term in equation (63), for M odd, depends on M, A, N and δ. Particular results are shown in Figs. 1 and 2 and a comparison with previous results is discussed in the next section.

To complete the solution, the solution for the first order system, from equation (55), can be written as,

$$\tilde{q}_{i1} = B_{i1}\cos\Omega_i\tau + B_{i2}\sin\Omega_i\tau + \frac{1}{2}\frac{b_{i11}}{\Omega_i^2} + \frac{b_{i11}}{2(4\Omega_1^2 - \Omega_i^2)}\cos 2\tilde{\Omega}_1\tau. \tag{65}$$

Using the initial conditions,

$$\tilde{q}_{i1} = 0 \quad \text{and} \quad \frac{\partial\tilde{q}_{i1}}{\partial\tau} = 0 \quad \text{for } \tau = 0 \tag{66}$$

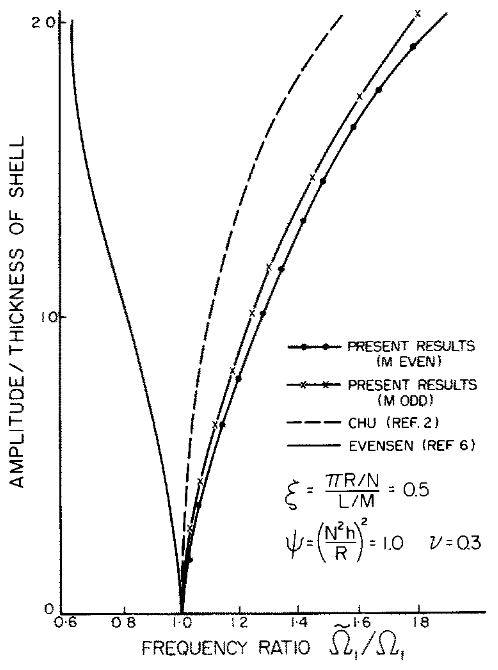

FIG. 1. Influence of large amplitude on natural frequency.

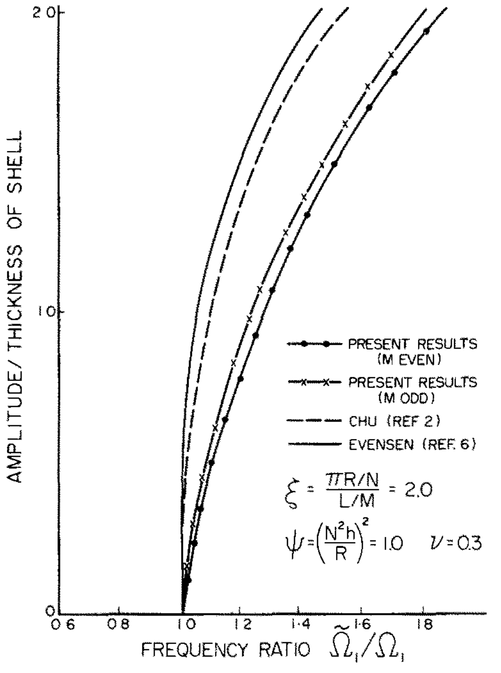

FIG. 2. Influence of large amplitude on natural frequency.

one can solve for B_{i1} and write

$$\bar{q}_{i1} = \frac{-2b_{i11}\Omega_1^2}{(4\Omega_1^2-\Omega_i^2)\Omega_i^2}\cos\Omega_i\tau+\frac{b_{i11}}{2\Omega_i^2}+\frac{b_{i11}}{2(4\Omega_1^2-\Omega_i^2)}\cos 2\tilde{\Omega}_1\tau. \tag{67}$$

From equation (67) it can be seen that because of the vanishing of the b_{211} in the present example, the second mode is not present in the solution at least up to the first order solution. By carrying on the higher order solution, it can be shown that the second mode is present in the second and higher order solutions because of the nonzero coupling coefficient C_{2211}. This is in contrast to the forced response case, as has been shown by Evensen [6], where the first and second mode responses are approximately of the same order when the forcing function corresponds to only one of the modes. Combining equations (24), (60), (67) and (59), one can write for the response subject to initial conditions, equations (57) and (58) as,

$$\tilde{w}(u,\beta,\tau) = \varepsilon\beta_1 \sin\tilde{\Omega}_1\tau \sin\dot{M}\alpha\cos\left(\frac{\dot{M}\beta}{\xi}\right)$$

$$+\varepsilon^2\frac{b_{311}}{\Omega_3^2}\left[-\frac{2\Omega_1^2}{4\Omega_1^2-\Omega_3^2}\cos\Omega_3\tau+\frac{1}{2}+\frac{\Omega_3^2}{2(4\Omega_1^2-\Omega_3^2)}\cos 2\tilde{\Omega}_1\tau\right]\sin\dot{M}\alpha. \tag{68}$$

COMPARISON WITH RESULTS BY OTHER AUTHORS

Chu [2] assumed a single mode of the form

$$W = A_{MN}(t)\sin\frac{\pi x}{L}\cos N\beta. \tag{69}$$

It can be seen that the above mode satisfies the simple-support boundary conditions, equation (6), on W. However, the condition of circumferential continuity of the circumferential inplane displacement v and the boundary condition on axial inplane displacement u were not satisfied. Thus, in Chu's formulation, the homogeneous part of the stress function, i.e. $\tilde{\Phi}_h$ as in equation (18), is not present. Whereas, in Nowinski's formulation [3], the mode shape was assumed as,

$$W(\chi, \beta, \tau) = A_{MN}\sin\frac{M\pi\chi}{L}\sin N\beta + f_0(t) \tag{70}$$

where the function $f_0(t)$ is determined in terms of A_{MN} using the condition of periodicity of the circumferential displacement; namely,

$$\int_0^{2\pi}\frac{\partial v}{\partial\beta}\,\mathrm{d}\beta = 0. \tag{71}$$

Thus,

$$f_0(t) = \frac{N^2}{8}A_{MN}^2. \tag{72}$$

It can be seen that the mode shape in equation (70) satisfies the simple-support condition $\partial^2 w/\partial x^2 = 0$, but the condition $w = 0$ is violated to the order $0\,(A_{MN}^2)$. Moreover, no condition was imposed on u and the homogeneous part of the stress function, $\tilde{\Phi}_h$, was taken as zero.

It should be pointed out that both Chu's [2] and Nowinski's [3] analysis involved a single mode and the final modal equation in both cases was of the form,

$$\frac{\partial^2 A_{MN}}{\partial\tau^2} + C_1 A_{MN} + C_2 A_{MN}^3 = 0 \tag{73}$$

which is of the Duffing type, involving only non-linear terms of static origin or the so-called "non-linear elasticity" terms and further C_2 was always positive. Thus, as can be seen from Figs. 1 and 2, the non-linearity effect on the vibrational frequency for all cases is of the hardening type, i.e. the frequency increases with amplitude.

On the other hand, Evensen's analysis [6] involved a two-mode representation of the type,

$$w(\chi, \beta, t) = A_{MN}(t)\sin\frac{M\pi\chi}{L}\cos N\beta + B_{MN}(t)\sin\frac{M\pi\chi}{L}\sin N\beta$$
$$+\frac{N^2}{4R}[A_{MN}^2 + B_{MN}^2]\sin^2\left(\frac{M\pi x}{L}\right). \tag{74}$$

It has been shown by Evensen [6], that the above mode shape satisfies the periodicity condition on v. However, even though the condition $w = 0$ is satisfied, the condition of vanishing moments, i.e. $\partial^2/\partial x^2 = 0$, is not satisfied at the simply-supported ends. Also no condition was imposed on the axial displacement u at the ends and further it was assumed that $\tilde{\Phi}_h = 0$. In non-dimensional form, Evensen's equations obtained from the Galerkin procedure, look like,

$$\frac{\partial^2 a_{MN}}{\partial \tau_{MN}^2} + a_{MN} + \frac{3}{8}\psi a_{MN}\left[a_{MN}\frac{\partial^2 a_{MN}}{\partial \tau_{MN}^2} + \left(\frac{\partial a_{MN}}{\partial \tau_{MN}}\right)^2 + b_{MN}\frac{\partial^2 b_{MN}}{\partial \tau_{MN}^2} + \left(\frac{\partial b_{MN}}{\partial \tau_{MN}}\right)^2\right]$$
$$+\psi\gamma a_{MN}[a_{MN}^2 + b_{MN}^2] + \psi\delta b_{MN}[a_{MN}^2 + b_{MN}^2]^2 = 0 \tag{75}$$

and another equation with A_{MN} and B_{MN} interchanged. In Evensen's notation,

$$\tau_{MN} = \Omega_{MN} t \tag{76}$$

$$\psi = \left(\frac{N^2 h}{R}\right)^2 \tag{77}$$

$$\xi = \frac{M\pi}{AN} \tag{78}$$

$$\Omega_{MN}^2 = \frac{E}{\rho_m R^2}\left[\frac{\xi^2}{(\xi^2+1)^2} + \psi\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right] \tag{79}$$

$$r = \xi^4\left[\frac{\psi}{12(1-\nu^2)} + \frac{1}{16} - \frac{1}{(\xi^2+1)^2}\right] \div \left[\frac{\xi^4}{(\xi^2+1)^2} + \psi\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right] \tag{80}$$

$$\delta = \frac{3\xi^4}{16}\left[\frac{1}{(\xi^2+1)^2} + \frac{1}{(9\xi^2+1)^2}\right] \div \left[\frac{\xi^4}{(\xi^2+1)^2} + \psi\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right]. \tag{81}$$

Thus, it should be observed, that Evensen's equations involve: (a) non-linear elasticity terms of order $0(A_{MN}^3)$, with the coefficients of non-linearity γ in equation (80) which can be either positive or negative depending on ξ, (b) non-linear elasticity terms of order $0(A_{MN}^5)$ with the coefficient of non-linearity δ in equation (81) which is always positive and in addition, (c) non-linear inertia terms of the type $f^2 f'' + f(f')^2$ (where $f \equiv A_{MN}$), with the coefficient of non-linearity, ψ, being always positive.

From general considerations of equations of the type (75), it can be shown (as, for instance, in Bolotin [7]), that: non-linear inertia terms with positive coefficient of non-linearity, in general, cause a decrease of natural frequency with amplitude; and non-linear elasticity terms either of third or fifth order with positive coefficients of non-linearity, in general, cause an increase of natural frequency with amplitude. Thus, in Evensen's analysis, the non-linear inertia terms always cause a decrease in natural frequency; the fifth order non-linear terms always cause an increase in natural frequency; and, on the other hand, third order non-linear terms cause a decrease in natural frequency, if

$$\frac{1}{(\xi^2+1)^2} > \frac{\psi}{12(1-\nu^2)} + \frac{1}{16} \tag{82}$$

and cause an increase in natural frequency, if

$$\frac{1}{(\xi^2+1)^2} < \frac{\psi}{12(1-\nu^2)} + \frac{1}{16}.$$

Thus, for results shown in Fig. 1; where $\psi = 1{\cdot}0$ and $\xi = 0{\cdot}5$ both the non-linear inertia terms and the third order non-linear elasticity terms are of the softening type, which dominate the hardening effect of the fifth order elasticity terms. On the other hand, for results shown in Fig. 2, where $\psi = 1{\cdot}0$ and $\xi = 2{\cdot}0$ both the third and fifth order non-linear elasticity terms cause a hardening effect, which dominate the softening effect of the non-linear inertia terms. Thus, Evensen's analysis appears to indicate that the nature of non-linearity essentially depends on the parameters ξ and ψ.

In contrast in the present analysis, the assumed mode shape, equation (14), satisfies both the simple-support conditions on w exactly, and the periodicity condition on v is satisfied only on the average, which has been found to involve approximations of order $0(A_{MN}^2)$. Also an "average" constraint condition is imposed on the inplane displacement u. Thus, the homogeneous solution $\tilde{\Phi}_h$ does not vanish and is given by equations (18)–(21). The modal equations involve only non-linear elasticity terms of second and third order. The primary third order non-linear coupling coefficients C_{iiii} and C_{iijj} for $i, j = 1, 2$ are always positive, where as the sign of the coefficient C_{33ii} for $i = 1, 2, 3$ depends on the values of the parameters M, A, δ and N. Also, when only one axial mode is retained, the second order coupling coefficients b_{ijk} vanish for M even, (whereas they are non-zero for M odd) denoting absence of second order coupling between the Nth and zeroth circumferential modes.

Thus, when M is even, in both Fig. 1 ($\xi = 0{\cdot}5$; $\psi = 1{\cdot}0$) and Fig. 2 ($\xi = 2{\cdot}0$, $\psi = 1{\cdot}0$) the coefficient C_{1111} is positive and hence a hardening type non-linearity is predicted and the hardening effect is found to be more dominant than the results given by Chu [2] and Nowinski [3]. When M is odd, $b_{ijk} \neq 0$ and are such that the coefficient γ_k in equation (63) is slightly decreased compared to the case when M is even and hence, as shown in Figs. 1 and 2, the hardening effect is less pronounced than the case when M is even.

The pronounced non-linear hardening effect in the present analysis as compared with the results of Chu [2] is due to the fact that the structure is effectively stiffer when the axial inplane displacements are prevented. The relative softening effect caused by the second order non-linear terms, in the present case for M odd, indicates the desirability of retaining additional axial modes in order to allow for the effect of circumferential coupling.

SUMMARY AND CONCLUSIONS

The non-linear flexural vibrations of a simply-supported thin-walled circular cylinder was analyzed by choosing mode shapes for the normal displacement and applying the Galerkin technique. The assumed mode shape satisfies the simple-support boundary conditions on normal displacement exactly and the inplane-displacement boundary conditions were satisfied on an average and a non-zero homogeneous solution was assumed for the inplane-stress function. Even though the explicit expressions for the Galerkin-type modal equations were given only a three-mode response case, a general solution procedure for the coupled non-linear response of $(M \times N)$ modes of the shell subjected to arbitrary initial conditions is presented. The theoretical results show, that because of the non-linear coupling, the response involves several modes, even though the shell is excited initially in a given mode. The relation between the natural frequency of each mode and their amplitudes is derived theoretically.

From the results of a three-mode response case, it has been found that the non-linearity was of the hardening type (for $\xi = 2$, $\psi = 1$; and for $\xi = 0{\cdot}5$ and $\psi = 1{\cdot}0$). When the axial wave number M is both even or odd.

The non-linearity, even though in qualitative agreement with that predicted by Chu [2] in all the cases considered here, is quantitatively stronger than that predicted by Chu and this difference can be attributed to the fact that for large values of ξ considered here (which implies that L/R of the cylinder is small) the imposition of the restraint condition on the inplane-displacement, as in present analysis, stiffens the structure considerably. The results, however, differ significantly from those of Evensen [6], whose analysis predicts that the non-linearity can be either the softening type (as in $\zeta = 0{\cdot}5$ and $\psi = 1{\cdot}0$ case) or of the hardening type (as in $\zeta = 2$ and $\psi = 1{\cdot}0$ case). A discussion of this discrepancy has been presented. Additional experiments, with emphasis on the boundary conditions, appear to be necessary to confirm the non-linear behavior of shell response for larger values of ζ.

It is also pointed out that when only one axial mode is retained, for M (axial wave number) odd, the Nth and zeroth circumferential modes are coupled in the quadratic non-linear terms in the modal equations while they are uncoupled for M even. For M odd, in both the cases, for which results are presented here, the effect of the quadratic coupling was to decrease the hardening effect of the cubic non-linear terms. This indicates the desirability of retaining additional axial modes for M even or odd to allow for the effect of circumferential coupling.

The present analysis is based on Donnell's shell equations which are valid for circumferential wave numbers n, such that $1/n^2 < 1$. The application of a more accurate shell theory such as due to Koiter [23] might prove useful.

Finally, it should be stressed the solution procedure given here, for free vibration problems represented by Galerkin-type of modal equations for many degrees-of-freedom and involving quadratic and cubic non-linear modal couplings, is general and can be applied to other vibrating structural elements like beams and plates.

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APPENDIX

Only the non-zero values of the coefficients Ω_i^2, b_{ijk}, C_{ijkl} used symbolically in the text are given here, for the three mode case, in terms of the already defined non-dimensional parameters characterizing the shell geometry and material properties.

$$\Omega_1^2 = C_1^{-1} + KC_2^{-1}C_1 \tag{A.1}$$

$$\Omega_2^2 = C_1^{-1} + KC_2^{-1}C_1 \tag{A.2}$$

$$\Omega_3^2 = C_2^{-1} + K + 2\nu^2 C_2^{\frac{1}{2}}\xi^3[1-(-1)^M]^2 \tag{A.3}$$

$$b_{113} = b_{223} = \left\{-4K\delta\frac{[1-(-1)^M]}{M\pi}[1 + C_1C_2^{-1} + C_2^{-1}(4C_2^{-0\cdot5} + C_3^{-0\cdot5})^2]\right.$$
$$\left. + \frac{KC_2^{-\frac{1}{2}}\nu\delta[(-1)^M - 1]}{(1-\nu^2)M\pi} - \frac{KC_3^{-\frac{1}{2}}\nu^2\delta[1-(-1)^M]}{M\pi(1-\nu^2)}\right\} \tag{A.4}$$

$$b_{311} = b_{322} = -\frac{1}{M\pi}[1-(-1)^M]\tfrac{2}{3}K\delta C_2^{-1}C_3^{-\frac{1}{2}}C_1 \tag{A.5}$$

$$b_{333} = +\frac{KA^2\nu^2}{(1-\nu^2)M\pi}[1-(-1)^M] - \frac{K\delta\nu}{(1-\nu^2)M\pi}C_2^{-\frac{1}{2}}[1-(-1)^M] \tag{A.6}$$

$$C_{1111} = C_{2222} = C_{2211} = C_{1122} = \frac{K\delta^2}{16}(C_2^{-1} + C_3^{-1}) + \frac{K\delta^2C_2^{-\frac{1}{2}}}{8(1-\nu^2)}(C_2^{-\frac{1}{2}} + C_3^{-\frac{1}{2}}\nu)$$
$$+ \frac{K\delta^2C_3^{-\frac{1}{2}}}{8(1-\nu^2)}(\nu C_2^{-\frac{1}{2}} + C_3^{-\frac{1}{2}}) \tag{A.7}$$

$$C_{1133} = C_{2233} = \left\{K\delta^2 C_2^{-\frac{1}{2}}\left[\tfrac{1}{2}+\tfrac{1}{4}C_3^{-1}(C_3^{-\frac{1}{2}}+4C_2^{-\frac{1}{2}})^{-2}\right] + \frac{C_2\delta^2 K}{4(1-\nu^2)} + \frac{KC_3^{-\frac{1}{2}}C_2^{-\frac{1}{2}}\delta^2}{4(1-\nu^2)}\right\} \tag{A.8}$$

$$C_{3333} = \left[-\frac{KA^2\delta\nu C_2^{-\frac{1}{2}}}{4(1-\nu^2)} + \frac{KC_2^{-1}\delta^2}{4(1-\nu^2)}\right]$$

and

$$C_{3311} = C_{3322} = \left\{-\frac{KA^2\delta}{8(1-\nu^2)}(\nu C_2^{-\frac{1}{2}}+C_3^{-\frac{1}{2}}) + \frac{KC_2^{-\frac{1}{2}}\delta^2(C_2^{-\frac{1}{2}}+\nu C_3^{-\frac{1}{2}})}{8(1-\nu^2)} + K\delta^2 C_2^{-\frac{1}{2}}\left[\tfrac{1}{4}+\tfrac{1}{8}C_3^{-1}(C_3^{-\frac{1}{2}}+4C_2^{-\frac{1}{2}})^{-2}\right]\right\}. \tag{A.9}$$

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Абстракт—Используя уравнения Донелла, исследуются нелинейные свободные колебания круглой цилиндрической оболочки. Используется модальное разложение для нормального перемещения, удовлетворящее точно граничным условиям для нормального перемещения, границуные же условия для перемещений в плоскости приблизительно удовлетворены методом усреднения. Используется метод Галеркина, с целью сведения задачи к системе сопряженных нелинейных обыкновенных дифференциальных уравнений для модальных амплитуд. Зти нелинейные дифференциальные уравнения решаются для произвольных начальных условий, путем приёма метода многовременного пересчёта. Даются явные значения коэффициентов, существуюших в выше упомнятой системе уравнений Галеркина. Они выражаются безразмерными параметрами, которые характеризуют геометрию оболочки и свойства материала, для трех видов колебаний, для которых представлены результаты для специальных начальных условий. Дается сравнение результатов с такими же, полученными в предыдущих исследования задачи и обсуждается разницы между результатами теорий.